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Frame-indifferent kinetic theory

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The second-order transport terms in a monatomic gas, originally derived by Burnett from Boltzmann's equation via the Chapman–Enskog iteration, have the curious property of being dependent on the observer's reference frame. However, frame-indifference has long been accepted as an essential property of constitutive equations in continuum mechanics. Various attempts have been made to find errors in the kinetic theory, but these have been countered by physical accounts of the frame-dependent terms that are independent of the Chapman–Enskog theory.

It is shown in this paper that both the Chapman–Enskog theory and the physical models are based on an inappropriate definition of the peculiar velocity. The Chapman–Enskog theory is easily corrected, and the result is a kinetic theory in harmony with the principle of material frame-indifference. A brief survey of the debate on this topic is presented, and corrected expressions are given for the Burnett terms.

1. Introduction

The generally accepted Chapman–Enskog theory, in which Boltzmann's equation is solved by an iterative process, has the defect of being frame-dependent, with the result that the expressions it gives for the pressure tensor $\boldsymbol{p}$ and the heat-flux vector $\boldsymbol{q}$ depend on the choice of the reference frame in which molecular velocities are specified. This frame-dependence appears first in the terms of second order in the Chapman–Enskog expansion parameter $\epsilon = \tau|\mathscr{D}\ln\phi|$, where τ is the collision interval, $\mathscr{D}$ is the material time derivative in 6-dimensional phase space, and ϕ is a typical macroscopic parameter like the pressure p, temperature T or the fluid velocity $\boldsymbol{v}$. Thus, in the expansions

$$\boldsymbol{p} = p\boldsymbol{I}+\boldsymbol{\pi}_1+\boldsymbol{\pi}_2+\dots,\quad \boldsymbol{q} = \boldsymbol{q}_1+\boldsymbol{q}_2=\dots,\quad \boldsymbol{\pi}_r = O(\epsilon^r),\quad \boldsymbol{q}_r = O(\epsilon^r), \tag{1.1}$$

whereas the classical first-order terms (for a monatomic gas)

$$\boldsymbol{\pi}_1 = -2\mu\overset{0}{\boldsymbol{\nabla}}\boldsymbol{v},\quad \boldsymbol{q}_1 = -k\nabla T,\quad \mu = p\tau_1,\quad k = \tfrac{5}{2}c_v\mu \tag{1.2}$$

are frame-indifferent,† as we shall show, the expressions for $\boldsymbol{\pi}_2$ and $\boldsymbol{q}_2$, usually associated with the name Burnett (1935), are not.

Under normal conditions the second-order terms $\boldsymbol{\pi}_2$ and $\boldsymbol{q}_2$ are only small corrections to $\boldsymbol{q}_1$ and $\boldsymbol{\pi}_1$, so their frame-dependence appears to be no more than a curious anomaly, with no practical significance. However, in a magnetoplasma it is in fact possible for terms $O(\epsilon^2)$ to be much more important than the corresponding

† Throughout this paper 'frame-indifferent' means 'not dependent on the motion of the *observer*'. In certain media rotation $\boldsymbol{\Omega}$ relative to the fixed stars can generate a microstructure (see §6) that alters the media properties. But provided these properties depend only on $\boldsymbol{\Omega}$, and not on $\boldsymbol{\Omega}-\boldsymbol{\Omega}_0$, where $\boldsymbol{\Omega}_0$ is the observer's angular velocity, they are 'frame-indifferent' on our definition.

first-order terms (Woods 1983), and in this case a frame-indifferent kinetic theory is essential.

There is also the general question of whether constitutive relations in continuum mechanics should be frame-indifferent or not. Before Müller (1972) drew attention to the frame-dependence of the relations for $\boldsymbol{\pi}_2$ and $\boldsymbol{q}_2$, it was generally accepted that correctly formulated constitutive relations were necessarily frame-indifferent (Eringen 1962; Truesdell & Noll 1965). Why, it may be asked, should the motion of the observer have any influence on the response of a continuum to imposed driving 'forces' like temperature gradients, boundary stresses, etc.? The conflict between standard kinetic theory and the principle of material frame-indifference has generated an interesting controversy that we shall review after outlining the Chapman–Enskog theory.

The objectives of this paper are to replace this theory by a frame-indifferent kinetic theory, to deduce corresponding expressions for $\boldsymbol{\pi}_2$ and $\boldsymbol{q}_2$, and in doing this to resolve the present conflict between kinetic theory and continuum mechanics.

That this conflict is substantive and not merely a question of different models admitting distinct solutions, can be seen as follows. Let ρ be the density and $\boldsymbol{F}$ be the non-collisional or body force acting on the molecules per unit mass, then, in a frame L, conservation of mass, momentum and energy gives

$$\mathrm{D}\rho+\rho\nabla\cdot\boldsymbol{v}=0, \quad \rho\mathrm{D}\boldsymbol{v}+\nabla\cdot\boldsymbol{p}=\rho\boldsymbol{F}, \tag{1.3a, b}$$

$$\rho\mathrm{D}(c_v T)+\boldsymbol{p}:\nabla\boldsymbol{v}=-\nabla\cdot\boldsymbol{q}, \quad \mathrm{D}\equiv\frac{\partial}{\partial t}+\boldsymbol{v}\cdot\nabla, \tag{1.3c, d}$$

D being the material time derivative in the fluid. These balance equations are frame-indifferent in the formal sense that corresponding terms of the equations in different frames must have the same physical significance.

Let primes denote quantities in a frame L' rotating with an angular velocity $\boldsymbol{\omega}$ relative to L, then in L' (1.3b) becomes

$$\rho\mathrm{D}'\boldsymbol{v}'+\nabla\cdot\boldsymbol{p}'=\rho\boldsymbol{F}',$$

provided that

$$\boldsymbol{F}'=\boldsymbol{F}-2\boldsymbol{\omega}\times\boldsymbol{v}'-\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r})+\nabla\cdot(\boldsymbol{p}'-\boldsymbol{p}) \tag{1.4}$$

at a point $\boldsymbol{r}$ measured from the axis of rotation. But $\boldsymbol{F}'$ is a body force, independent of the nature of the medium, only if $\nabla\cdot(\boldsymbol{p}'-\boldsymbol{p})=0$. Similarly, as the (scalar) terms in (1.3c) represent distinct physical processes, independent of the observer's motion, they have the same values in L' as in L, i.e. $\nabla\cdot\boldsymbol{q}=\nabla\cdot\boldsymbol{q}'$ and $\boldsymbol{p}':\nabla\boldsymbol{v}'=\boldsymbol{p}:\nabla\boldsymbol{v}$. Moreover, as $\boldsymbol{p}$ and $\boldsymbol{p}'$ are symmetric, $\boldsymbol{p}':\nabla\boldsymbol{v}'=\boldsymbol{p}':(\nabla\boldsymbol{v}')^{\mathrm{s}}=\boldsymbol{p}':(\nabla\boldsymbol{v})^{\mathrm{s}}=\boldsymbol{p}:(\nabla\boldsymbol{v})^{\mathrm{s}}$, so that $\boldsymbol{p}'=\boldsymbol{p}$. Thus any method of obtaining constitutive relations for $\boldsymbol{p}$ and $\boldsymbol{q}$ should entail

$$\boldsymbol{p}'=\boldsymbol{p}, \quad \nabla\cdot\boldsymbol{q}'=\nabla\cdot\boldsymbol{q}. \tag{1.5}$$

The second of these physical constraints can be reduced to $\boldsymbol{q}'=\boldsymbol{q}$ by considering the equation for entropy change, but we shall not pursue this.

2. The Chapman–Enskog kinetic theory

A brief outline of the Chapman–Enskog theory (see Chapman & Cowling 1970) will enable us to detect the point at which frame-dependence is tacitly introduced.

Let $f(\boldsymbol{r},\boldsymbol{w},t)\,\mathrm{d}\boldsymbol{r}\,\mathrm{d}\boldsymbol{w}$ be the number of molecules lying in the element $\mathrm{d}\boldsymbol{r}\,\mathrm{d}\boldsymbol{w}$ of

6-dimensional phase space, then, provided the body force $\boldsymbol{F}$ is independent of the molecular velocity $\boldsymbol{w}$, the distribution function f satisfies the kinetic equation

$$\frac{\partial f}{\partial t}+\boldsymbol{w}\cdot\frac{\partial f}{\partial \boldsymbol{r}}+\boldsymbol{F}\cdot\frac{\partial f}{\partial \boldsymbol{w}}=\frac{1}{\tau}\mathbb{C}(ff), \tag{2.1}$$

where $\mathbb{C}/\tau$ is the rate of change due to particle collisions, τ being the collision interval. The average of $\boldsymbol{w}$ over velocity space is the fluid velocity $\boldsymbol{v}$, i.e.

$$\boldsymbol{v}(\boldsymbol{r},t)=\frac{1}{n}\int \boldsymbol{w}f\,\mathrm{d}\boldsymbol{w}, \tag{2.2}$$

where $n(\boldsymbol{r},t)$ is the number density. We now introduce the 'peculiar' velocity

$$\boldsymbol{c}=\boldsymbol{w}-\boldsymbol{v}(\boldsymbol{r},t), \tag{2.3}$$

replace $f(\boldsymbol{r},\boldsymbol{w},t)$ by $f(\boldsymbol{r},\boldsymbol{c},t)$, and write (2.1) in the form

$$\mathbb{C}(ff)=\tau\mathscr{D}f, \tag{2.4}$$

where

$$\mathscr{D}\equiv \mathrm{D}+\boldsymbol{c}\cdot\boldsymbol{\nabla}+(\boldsymbol{F}-\mathrm{D}\boldsymbol{v}-\boldsymbol{c}\cdot\boldsymbol{e})\cdot\frac{\partial}{\partial \boldsymbol{c}},\quad \boldsymbol{e}\equiv\boldsymbol{\nabla}\boldsymbol{v}, \tag{2.5}$$

and D is the derivative defined in (1.3).

It is now assumed that $\tau\mathscr{D}\ln f=0(\epsilon)\ll 1$. To zero order in ϵ, (2.4) gives $\mathbb{C}(f_0f_0)=0$, where, by the usual arguments, the equilibrium distribution f_0 is

$$f_0=n\left(\frac{m}{2\pi\kappa T}\right)^{\frac{3}{2}}\mathrm{e}^{-w^2},\quad \boldsymbol{w}\equiv\frac{\boldsymbol{c}}{C},\quad C\equiv\left(\frac{2\kappa T}{m}\right)^{\frac{1}{2}}. \tag{2.6}$$

The next step is to substitute the expansions

$$f=f_0+f_1+f_2+\dots,\quad \mathscr{D}=\mathscr{D}_0+\mathscr{D}_1+\mathscr{D}_2+\dots,\quad f_r=O(\epsilon^r),\quad \tau\mathscr{D}_r=O(\epsilon^{r+1}), \tag{2.7}$$

into (2.4) to obtain the sequence of equations

$$\Phi_0=0,\quad \Phi_1=-\tau\mathscr{D}_0\ln f_0,\quad \Phi_2=-\frac{\tau\mathscr{D}_0f_1}{f_0}-\tau\mathscr{D}_1\ln f_0,\quad \dots, \tag{2.8}$$

where

$$\Phi_r=-f_0\mathbb{C}(f_0f_r+f_1f_{r-1}+\dots+f_rf_0)=\Phi_r(f_0,f_1,\dots,f_r). \tag{2.9}$$

Equations (2.8) are solved in turn for $f_1, f_2, \dots$.

That $\mathscr{D}$ needs to be expanded is evident from (1.1) and (1.3), which yield

$$\left.\begin{aligned}\mathrm{D}_0n&=-n\boldsymbol{\nabla}\cdot\boldsymbol{v},\quad \mathrm{D}_rn=0,\\ \rho\mathrm{D}_0\boldsymbol{v}&=-\boldsymbol{\nabla}p+\rho\boldsymbol{F},\quad \rho\mathrm{D}_r\boldsymbol{v}=-\boldsymbol{\nabla}\cdot\boldsymbol{\pi}_r\quad(r\geqslant 1),\\ \rho c_v\mathrm{D}_0T&=-p\boldsymbol{\nabla}\cdot\boldsymbol{v},\quad \rho c_v\mathrm{D}_rT=-\boldsymbol{\pi}_r:\boldsymbol{\nabla}\boldsymbol{v}-\boldsymbol{\nabla}\cdot\boldsymbol{q}_r,\end{aligned}\right\} \tag{2.10}$$

where† D_r is $O(\epsilon^r)$. Hence from (2.5)

$$\mathscr{D}_0=D_0+\boldsymbol{c}\cdot\boldsymbol{\nabla}+\left(\frac{\boldsymbol{\nabla}p}{\rho}-\boldsymbol{c}\cdot\boldsymbol{e}\right)\cdot\frac{\partial}{\partial\boldsymbol{c}},\quad \mathscr{D}_r=\mathrm{D}_r+\frac{1}{\rho}\boldsymbol{\nabla}\cdot\boldsymbol{\pi}_r\cdot\frac{\partial}{\partial\boldsymbol{c}}\quad(r\geqslant 1). \tag{2.11}$$

The only higher-order derivative required later is $\mathscr{D}_1\ln f_0$. From (2.6), (2.10) and (2.11) this is

$$\begin{aligned}\mathscr{D}_1\ln f_0&=\mathrm{D}_1\ln n+(w^2-\tfrac{3}{2})\,\mathrm{D}_1\ln T-2C^{-2}\boldsymbol{c}\cdot\mathscr{D}_1\boldsymbol{c}\\ &=-\frac{1}{p}\{(\tfrac{2}{3}w^2-1)(\boldsymbol{\pi}_1:\boldsymbol{\nabla}\boldsymbol{v}+\boldsymbol{\nabla}\cdot\boldsymbol{q}_1)+\boldsymbol{\nabla}\cdot\boldsymbol{\pi}_1\cdot\boldsymbol{c}\}.\end{aligned} \tag{2.12}$$

Let

$$\boldsymbol{g}\equiv C\boldsymbol{\nabla}\ln p,\quad \boldsymbol{h}\equiv C\boldsymbol{\nabla}\ln T,\quad \overset{\times}{e}\equiv\boldsymbol{\nabla}\cdot\boldsymbol{v}, \tag{2.13}$$

† In Chapman & Cowling's text the order indices are attached to $\partial/\partial t$ rather than to D, but the change introduced in (2.10) makes no essential difference to the final expressions.

then from (2.10) and (2.11)

$$\mathscr{D}_0 \boldsymbol{w} = \tfrac{1}{2}\boldsymbol{g} - \tfrac{1}{2}\boldsymbol{w}\boldsymbol{w}\cdot\boldsymbol{h} - \boldsymbol{w}\cdot(\boldsymbol{e} - \tfrac{1}{3}\overset{\times}{e}\boldsymbol{I}), \tag{2.14a}$$

$$\mathscr{D}_0 \ln p = \boldsymbol{w}\cdot\boldsymbol{g} - \tfrac{5}{3}\overset{\times}{e}, \quad \mathscr{D}_0 \ln T = \boldsymbol{w}\cdot\boldsymbol{h} - \tfrac{2}{3}\overset{\times}{e}. \tag{2.14b}$$

By $p = knT$ and (2.6), $\ln f_0 = \ln p - \frac{5}{2}\ln T - w^2 + \text{const}$, whence

$$\mathscr{D}_0 \ln f_0 = (w^2 - \tfrac{5}{2})\, \boldsymbol{w}\cdot\boldsymbol{h} + 2\boldsymbol{w}\boldsymbol{w} : \overset{0}{\boldsymbol{e}}. \tag{2.15}$$

where $\overset{0}{\boldsymbol{e}}$ is the deviator of $\boldsymbol{e}$ (symmetrical part from which the trace has been removed).

When (2.8) have been solved for f_r, $r = 1, 2, \dots$, the heat-flux vector $\boldsymbol{q} = \boldsymbol{q}_1 + \boldsymbol{q}_2 + \dots$ and the pressure tensor $\boldsymbol{p} = p\boldsymbol{I} + \boldsymbol{\pi}_1 + \boldsymbol{\pi}_2 + \dots$, can be found from

$$\boldsymbol{q}_r = \tfrac{1}{2}m \int c^2 \boldsymbol{c} f_r \,\mathrm{d}\boldsymbol{c} = pC \frac{4}{\pi^{\frac{1}{2}}} \int_\Omega \int_0^\infty w^5 \mathrm{e}^{-w^2} \phi_r \,\mathrm{d}w\, \hat{\boldsymbol{w}}\, \mathrm{d}\Omega, \tag{2.16}$$

$$\boldsymbol{\pi}_r = m \int \boldsymbol{c}\boldsymbol{c} f_r \,\mathrm{d}\boldsymbol{c} = 2p \frac{4}{\pi^{\frac{1}{2}}} \int_\Omega \int_0^\infty w^4 \mathrm{e}^{-w^2} \phi_r \,\mathrm{d}w\, \hat{\boldsymbol{w}}\hat{\boldsymbol{w}}\, \mathrm{d}\Omega, \tag{2.17}$$

where

$$\phi_r \equiv f_r / f_0, \tag{2.18}$$

$\hat{\boldsymbol{w}} = \boldsymbol{w}/w$, $4\pi\,\mathrm{d}\Omega$ is the element of solid angle, and the integrands follow from (2.6).

To make further progress, we need expressions for the collision operators $\boldsymbol{\Phi}_r$ defined in (2.9). The Boltzmann collision operator is complicated, and as its frame-indifference is not in question (Wang 1975), we shall adopt the BGK relaxation model, $\mathbb{C} = f_0 - f = -(f_1 + f_2 + \dots)$, for which (2.9) and (2.18) give

$$\boldsymbol{\Phi}_r = \phi_r \quad (r \geqslant 1). \tag{2.19}$$

As we shall see, this is remarkably accurate provided one *ad hoc* modification is adopted.

3. The Burnett terms

We shall now calculate the Burnett expressions for $\boldsymbol{\pi}_2$ and $\boldsymbol{q}_2$, so that the effect of the corrections we shall make later to remove their frame-dependence is easily traced. It is also useful to have a simple account of the derivation of $\boldsymbol{\pi}_2$ and $\boldsymbol{q}_2$ to replace the complicated treatment (albeit accurate for the frame-indifferent terms) given in advanced texts on kinetic theory (Chapman & Cowling 1970; Fertziger & Kaper 1972).

The relaxation collision operator introduced in (2.19) has one minor disadvantage, namely that it gives unity for the Prandtl number, $P_r = \mu C_p / k$ instead of the correct value, which in a neutral monatomic gas is $\frac{2}{3}$. While this can be overcome by allowing the collision interval τ to depend on $\boldsymbol{c}$, in order to conserve the number, momentum and energy of particles at collisions, it is then necessary to complicate (2.19) somewhat. We shall not do this, but will follow a different method that is quite accurate, despite appearing to be *ad hoc*. The correct Prandtl number is obtained by adopting two distinct collision intervals, say τ_1 for momentum transport, and τ_2 for energy transport, related by

$$\tau_1 = P_r \tau_2 = \tfrac{2}{3}\tau_2, \tag{3.1}$$

and adopting the rule:

$$\text{For } \boldsymbol{q} \text{ use} \quad \tau = \tau_2; \quad \text{for } \boldsymbol{p} \text{ use} \quad \tau = \tau_1. \tag{3.2}$$

With this modification the relaxation model gives exact values for all eleven coefficients in Burnett's second-order theory.

The first-order theory follows from (2.8), (2.15)–(2.18) and (3.2). We find

$$\phi_1 = -\tau_2(w^2-\tfrac{5}{2})\,\boldsymbol{w}\cdot\boldsymbol{h} - 2\tau_1\,\boldsymbol{ww}:\overset{0}{\boldsymbol{e}}, \tag{3.3}$$

$$\boldsymbol{\pi}_1 = -2\mu\overset{0}{\boldsymbol{\nabla}\boldsymbol{v}}, \quad \boldsymbol{q}_1 = -\kappa\boldsymbol{\nabla}T, \tag{3.4}$$

where

$$\mu = p\tau_1, \quad \kappa = \frac{5}{2}\frac{k}{m}p\tau_2 = \tfrac{5}{2}c_v\mu. \tag{3.5}$$

By (2.8) and (2.18) second-order theory is based on

$$\phi_2 = -\tau\{\mathscr{D}_0\phi_1 + \phi_1\mathscr{D}_0\ln f_0 + \mathscr{D}_1\ln f_0\}. \tag{3.6}$$

If it is assumed that molecules a distance r apart repel with a force proportional to $r^{-\nu}$, the usual dimensional analysis yields

$$p\tau_1 = \mu = AT^s, \quad s = \tfrac{1}{2} + \frac{2}{\nu-1}, \tag{3.7}$$

where A and s are constants. It follows from (2.14) that

$$\mathscr{D}_0\tau_1 = \tau_1\mathscr{D}_0(s\ln T - \ln p) = \tau_1(s\boldsymbol{w}\cdot\boldsymbol{h} - \boldsymbol{w}\cdot\boldsymbol{g} + \tfrac{1}{3}\overset{\times}{e}(5-2s)), \tag{3.8}$$

and similarly for $\mathscr{D}_0\tau_2$. From (2.12), (3.4) and (3.8) we get

$$\mathscr{D}_1\ln f_0 = 2\tau_1\{(\tfrac{2}{3}w^2-1)\,\overset{0}{\boldsymbol{e}}:\boldsymbol{e} + (\boldsymbol{\nabla}\cdot\overset{0}{\boldsymbol{e}} + s\overset{0}{\boldsymbol{e}}\cdot\boldsymbol{\nabla}\ln T)\cdot\boldsymbol{c}\} + \frac{5}{2}\frac{k}{m}\tau_2(\tfrac{2}{3}w^2-1)\,(\nabla^2 T + s\boldsymbol{\nabla}T\cdot\boldsymbol{\nabla}\ln T). \tag{3.9}$$

To calculate $\mathscr{D}_0\phi_1$ we use (2.14), (3.3) and

$$\mathscr{D}_0\boldsymbol{h} = \mathscr{D}_0\boldsymbol{h} + \boldsymbol{c}\cdot\boldsymbol{\nabla}\boldsymbol{h}, \quad \mathscr{D}_0\overset{0}{\boldsymbol{e}} = \boldsymbol{D}_0\overset{0}{\boldsymbol{e}} + \boldsymbol{c}\cdot\boldsymbol{\nabla}\overset{0}{\boldsymbol{e}}. \tag{3.10}$$

Substituting these expressions into (3.6), we arrive at a value for ϕ_2 that can be arranged as

$$\begin{aligned}
\phi_1 = \tau\Big\{&\tfrac{2}{3}\tau_2(\tfrac{7}{2}-s)\,(w^2-\tfrac{5}{2})\,\overset{\times}{e}\,\boldsymbol{w}\cdot\boldsymbol{h} + \tau_2\frac{C}{T}(w^2-\tfrac{5}{2})\,\boldsymbol{w}\cdot(\mathrm{D}\boldsymbol{\nabla}T - \boldsymbol{e}\cdot\boldsymbol{\nabla}T)\\
&+ 2\tau_1\boldsymbol{g}\cdot\overset{0}{\boldsymbol{e}}\cdot\boldsymbol{w} - 2\tau_1\boldsymbol{g}\cdot\boldsymbol{www}:\overset{0}{\boldsymbol{e}} + 2[(\tau_1+\tau_2)\,(w^2-\tfrac{7}{2}) + \tau_1 s]\,\boldsymbol{h}\cdot\boldsymbol{www}:\overset{0}{\boldsymbol{e}}\\
&- 2\tau_1 s\boldsymbol{h}\cdot\overset{0}{\boldsymbol{e}}\cdot\boldsymbol{w} + 2\tau_1 C(\boldsymbol{www} \,\vdots\, \boldsymbol{\nabla}\overset{0}{\boldsymbol{e}} - \boldsymbol{\nabla}\cdot\overset{0}{\boldsymbol{e}}\cdot\boldsymbol{w})\Big\}\\
+\tau\Big\{&\tfrac{4}{3}\tau_1(\tfrac{7}{2}-s)\,\overset{\times}{e}\,\boldsymbol{ww}:\overset{0}{\boldsymbol{e}} + 2\tau_1\,\boldsymbol{w}\cdot(\mathrm{D}\overset{0}{\boldsymbol{e}} - 2\boldsymbol{e}\cdot\overset{0}{\boldsymbol{e}})\cdot\boldsymbol{w} + \frac{2k}{m}\tau_2[(w^2-\tfrac{5}{2})\,\boldsymbol{w}\cdot\boldsymbol{\nabla\nabla}T\cdot\boldsymbol{w}\\
&- \tfrac{5}{4}(\tfrac{2}{3}w^2-1)\,\nabla^2 T] + \tfrac{1}{2}\tau_2(w^2-\tfrac{5}{2})\,\boldsymbol{g}\cdot\boldsymbol{h} - \tau_2(w^2-\tfrac{7}{2})\,\boldsymbol{g}\cdot\boldsymbol{ww}\cdot\boldsymbol{h}\\
&- \frac{k}{m}\tau_2(w^2-\tfrac{5}{2}+\tfrac{5}{2}s)\,\boldsymbol{\nabla}T\cdot\boldsymbol{\nabla}\ln T + 2\frac{k}{m}\tau_2[(w^2-\tfrac{5}{2})\,(w^2-\tfrac{7}{2}+s) - w^2]\,\boldsymbol{\nabla}T\cdot\boldsymbol{ww}\cdot\boldsymbol{\nabla}\ln T\\
&+ 4\tau_1\,\overset{0}{\boldsymbol{e}}:\boldsymbol{wwww}:\overset{0}{\boldsymbol{e}} - 2\tau_1(\tfrac{2}{3}w^2-1)\,\overset{0}{\boldsymbol{e}}:\overset{0}{\boldsymbol{e}}\Big\},
\end{aligned} \tag{3.11}$$

where the first term contributes only to $\boldsymbol{q}_2$ and the second only to $\boldsymbol{\pi}_2$. Thus by (3.2) we put the first τ equal to τ_2 and the second equal to τ_1.

Now substituting (3.11) into (2.16) and (2.17) we arrive at the Burnett equations

$$\boldsymbol{q}_2 = \frac{k}{m} p\tau_1^2 \left\{ \theta_1 \nabla\cdot\boldsymbol{v}\nabla T + \theta_2(\mathrm{D}\nabla T - \boldsymbol{e}\cdot\nabla T) + \theta_3 \frac{T}{p}\nabla p\cdot\overset{0}{\boldsymbol{e}} + \theta_4 T\nabla\cdot\overset{0}{\boldsymbol{e}} + 3\theta_5 \nabla T\cdot\overset{0}{\boldsymbol{e}} \right\}, \tag{3.12}$$

with

$$\theta_1 = \tfrac{15}{4}(\tfrac{7}{2}-s), \quad \theta_2 = \tfrac{45}{8}, \quad \theta_3 = -3, \quad \theta_4 = 3, \quad \theta_5 = \tfrac{35}{4}+s,$$

$$\boldsymbol{\pi}^2 = p\tau_1^2 \left\{ \varpi_1 \nabla\cdot\boldsymbol{v}\overset{0}{\boldsymbol{e}} + \varpi_2(D\overset{0}{\boldsymbol{e}} - \overset{0}{\overline{2\boldsymbol{e}\cdot\overset{0}{\boldsymbol{e}}}}) + \varpi_3 \frac{k}{m}\overset{0}{\overline{\nabla\nabla T}} + \varpi_4 \frac{1}{pT}\overset{0}{\overline{\nabla p\nabla T}} + \varpi_5 \frac{k}{mT}\overset{0}{\overline{\nabla T\nabla T}} + \varpi_6 \overset{0}{\overline{\overset{0}{\boldsymbol{e}}\cdot\overset{0}{\boldsymbol{e}}}} \right\} \tag{3.13}$$

with

$$\varpi_1 = \tfrac{4}{3}(\tfrac{7}{2}-s), \quad \varpi_2 = 2, \quad \varpi_3 = 3, \quad \varpi_4 = 0, \quad \varpi_5 = 3s, \quad \varpi_6 = 8.$$

All of the above θ- and ϖ-coefficients agree exactly with those obtained from the Boltzmann integral. In a private communication (1978) Professor T. G. Cowling showed that the relaxation model gives exact results for a Maxwellian gas; i.e. he obtained (3.12) and (3.13) for the special case $s = 1$.

But according to the arguments presented in § 1, the above expressions are wrong, since, by

$$\boldsymbol{e} = \overset{0}{\boldsymbol{e}} + \tfrac{1}{3}\overset{\times}{e}\boldsymbol{I} - \boldsymbol{\Omega}\times\boldsymbol{I}, \quad \boldsymbol{\Omega} \equiv \tfrac{1}{2}\nabla\times\boldsymbol{v}, \tag{3.14}$$

the terms $\mathrm{D}\nabla T - \boldsymbol{e}\cdot\nabla T$ and $\mathrm{D}\overset{0}{\boldsymbol{e}} - \overset{0}{\overline{2\boldsymbol{e}\cdot\overset{0}{\boldsymbol{e}}}}$ contain the frame-dependent parts

$$\boldsymbol{I} = \mathrm{D}\nabla T + \boldsymbol{\Omega}\times\nabla T, \quad \boldsymbol{J} = \mathrm{D}\overset{0}{\boldsymbol{e}} + 2\boldsymbol{\Omega}\times\overset{0}{\boldsymbol{e}}. \tag{3.15}$$

For let $\boldsymbol{I}$ have the value $\boldsymbol{I}'$ in a frame L' rotating with angular velocity $\boldsymbol{\omega}$ relative to the frame L in which (3.15) hold, then

$$\boldsymbol{I}' = \mathrm{D}'\nabla T + \boldsymbol{\Omega}'\times\nabla T = \mathrm{D}\nabla T - \boldsymbol{\omega}\times\nabla T + (\boldsymbol{\Omega}-\boldsymbol{\omega})\times\nabla T = \boldsymbol{I} - 2\boldsymbol{\omega}\times\nabla T; \tag{3.16}$$

similarly $\boldsymbol{J}' = \boldsymbol{J} - 4\boldsymbol{\omega}\times\overset{0}{\boldsymbol{e}}$. It follows from (3.12) and (3.13) that

$$\boldsymbol{p}' = \boldsymbol{p} + 4p\tau_1^2\varpi_2\boldsymbol{\omega}\times\overset{0}{\boldsymbol{e}}, \tag{3.17a}$$

$$\nabla\cdot\boldsymbol{q}' = \nabla\cdot\boldsymbol{q} + \boldsymbol{\omega}\times\nabla T\cdot\nabla\beta, \quad \beta = 2\theta_2\frac{k}{m}p\tau_1^2, \tag{3.17b}$$

in disagreement with (1.5). To have frame-indifferent expressions in (3.15) it is necessary to change the sign of $\boldsymbol{\Omega}$.

We are now in a position to discuss the several attempts to resolve the paradox posed by the θ_2 and ϖ_2 terms in (3.12) and (3.13).

4. Some explanations of the frame-dependence paradox

As it is possible to derive the Burnett equations from a generalized mean-free-path model (Woods 1979), to resolve the paradox it is not sufficient to fault the Chapman–Enskog iterative solution of Boltzmann's equation. The explanation must also dispose of apparently sound physical arguments yielding terms of the type given in (3.15).

One such argument is the following. Adding $\boldsymbol{q}_1$ to (3.12) and using (3.4) and (3.5), we find

$$\boldsymbol{q} = \boldsymbol{q}_1 + \boldsymbol{q}_2 = \boldsymbol{q}_1 - \tau_2\mathrm{D}\boldsymbol{q}_1 - \tau_2\boldsymbol{\Omega}\times\boldsymbol{q}_1 + \dots, \tag{4.1}$$

where the terms omitted are frame-indifferent. The time derivative in (4.1) is due to the small delay of τ_2 in the molecular transport of energy to $P(\boldsymbol{r}, t)$ from points a mean

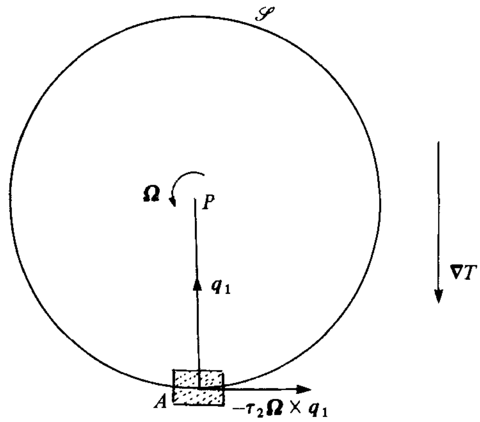

FIGURE 1. Microconvection of heat.

free path λ away. The molecules experience their last collision on a sphere $\mathscr{S}$ of radius λ and then move to the centre P (see figure 1). The fluid particle centred on P rotates about it, as if in a rigid-body motion with angular velocity $\boldsymbol{\Omega} = \frac{1}{2}\nabla \times \boldsymbol{v}$. Thus, during the time it takes the molecules to travel from some point A on $\mathscr{S}$ to P, the fluid motion convects some energy in a direction parallel to $-\boldsymbol{\Omega} \times \boldsymbol{q}_1$. And, as τ_2 is the relaxation time for the molecules to lose their excess energy, this convection (relative to P) adds $-\tau_2 \boldsymbol{\Omega} \times \boldsymbol{q}_1$ to $\boldsymbol{q}$, in agreement with (4.1).

Müller (1972) opened the debate by pointing out that the Ikenberry–Truesdell (1956) iterative procedure gave frame-dependent expressions for $\boldsymbol{q}$ and $\boldsymbol{p}$ for a gas of Maxwellian molecules. In particular he showed that the heat conduction in a rigidly rotating gas depends on the angular velocity of the rotation (cf. (3.17)) a frame-dependence that he attributed to the Coriolis force. Edelen & McLennan (1973) followed this lead by showing that the Burnett formulae were also frame-dependent; they concluded that frame-indifference was not a principle of continuum mechanics, but was merely a 'convenient' rule (albeit sometimes wrong!).

Wang (1975, 1976) opposed these conclusions, with the vague argument that mathematical uncertainties of various types in the Chapman–Enskog expansion used by Burnett left it an open question as to whether his formula undermined frame-indifference or not. Speziale (1981) continued this line, with the claim that the iterative procedure used by Chapman and Enskog was at fault, specifically that their assumption that $\partial \boldsymbol{v}/\partial t$ did not depend explicitly on t was frame-dependent, and hence led to values of $\boldsymbol{q}$ and $\boldsymbol{\pi}$ similarly defective. But the Chapman–Enskog treatment of $\partial \boldsymbol{v}/\partial t$ is no more than equivalent to eliminating $\mathrm{D}\boldsymbol{v} - \boldsymbol{F}$ from (2.5) by using the equation of motion, and, as $\mathrm{D}_0 \boldsymbol{v} - \boldsymbol{F}, \mathrm{D}_1 \boldsymbol{v}, \mathrm{D}_2 \boldsymbol{v}, \ldots$ are each frame-indifferent, this is obviously not the source of the problem. Truesdell & Muncaster (1980) considered that the principle of frame-indifference might be saved by even higher-order terms neglected in the Burnett theory, although how a cancellation between frame-dependent terms of different differential order could happen was not explained.

On the other side of the debate, Söderholm (1976) advanced physical arguments, based on Müller's observation about the Coriolis force, to demonstrate that heat conduction in a gas should have the Burnett frame-dependence. And, as his arguments did not invoke the Chapman–Enskog theory, it followed that Wang's strictures on Burnett's method were beside the point.

Prior to writing this paper, the author's view (Woods 1982*a*) was that, while all the irreversible terms in constitutive relations were certainly frame-indifferent – the associated dissipation could not depend on the observer – this did not necessarily apply to all of the reversible terms. Now the Burnett (second-order) terms in $\boldsymbol{q}$ and $\boldsymbol{p}$ are reversible (Woods 1982*a*), so they do not contribute to the dissipation, and hence (it was argued) do not need to be frame-indifferent. Noting that there were at least three distinct methods of obtaining the Burnett equations, i.e. from kinetic theory (Burnett 1935), from thermodynamic arguments (Woods 1980), and from a generalized mean-free-path theory (Woods 1979), the author accepted that these were correct and hence that frame-indifference applied with certainty only to the irreversible terms. However, in the light of the following correction of the Chapman–Enskog theory, this conclusion may need revision.

5. Frame-indifferent kinetic theory

The transformation of the independent velocity variable from $\boldsymbol{w}$ to $\boldsymbol{c}$ is not fully specified by

$$\boldsymbol{w} = \boldsymbol{c} + \boldsymbol{v}(\boldsymbol{r}, t), \tag{5.1}$$

for, as the differentials $\mathrm{d}\boldsymbol{w}$ and $\mathrm{d}\boldsymbol{c}$ are required in this transformation, it is also necessary to specify the relation between the reference frames L in which $\boldsymbol{w}$ is measured, and O in which $\boldsymbol{c}$ is measured. Equation (2.4) is derived from (2.1) on the tacit assumption that O remains in a fixed orientation relative to L, whereas, were O embedded in a fluid element, it would rotate relatively to L with an angular velocity $\boldsymbol{\Omega} = \frac{1}{2}\nabla \times \boldsymbol{v}$. There is no justification for maintaining O fixed relative to L, and in the author's view the peculiar velocity $\boldsymbol{c}$ of a given molecule must be defined in a frame O that is convected in *both* position and orientation with the fluid particle – any other choice would be non-unique.

Referring to figure 1, we see that this change means that the conduction is to be measured in a frame that carries AB into $A'B'$ after time τ_2. In this case the energy transport $-\tau_2 \boldsymbol{\Omega} \times \boldsymbol{q}_1$ does not form part of the *conducted* heat, but the rotation alters the effective direction of $\boldsymbol{q}$ from AB to $A'B'$, giving

$$\boldsymbol{q} = \boldsymbol{q}_1 - \tau_2 \mathrm{D}\boldsymbol{q}_1 + \tau_2 \boldsymbol{\Omega} \times \boldsymbol{q}_1 + \dots. \tag{5.2}$$

This argument is not conclusive, but it shows that the physical description attached to figure 1 is based on a definition of $\boldsymbol{c}$ that gratuitously introduces frame-dependence into heat flux.

Let $\boldsymbol{a}(\boldsymbol{r}, \boldsymbol{w}, t)$ be a vector specified in the L-frame and let $\boldsymbol{A}(\boldsymbol{r}, \boldsymbol{c}, t)$ be the same vector specified in the fully convected O-frame, i.e. a frame that rotates with angular velocity $\boldsymbol{\Omega}$ relative to L. Denote differentials in the O-frame by d_0; then by (5.1)

$$\mathrm{d}\boldsymbol{w} = \mathrm{d}_0\boldsymbol{c} + \frac{\partial \boldsymbol{v}}{\partial t}\mathrm{d}t + \mathrm{d}\boldsymbol{r}\cdot\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{r}} = \mathrm{d}\boldsymbol{c} + \left(\frac{\partial \boldsymbol{v}}{\partial t} - \boldsymbol{\Omega}\times\boldsymbol{c}\right)\mathrm{d}t + \mathrm{d}\boldsymbol{r}\cdot\boldsymbol{e}, \quad \boldsymbol{e} \equiv \nabla\boldsymbol{v},$$

so that

$$\mathrm{d}\boldsymbol{a} = \frac{\partial \boldsymbol{a}}{\partial t}\mathrm{d}t + \mathrm{d}\boldsymbol{r}\cdot\frac{\partial \boldsymbol{a}}{\partial \boldsymbol{r}} + \left\{\mathrm{d}\boldsymbol{c} + \left(\frac{\partial \boldsymbol{v}}{\partial t} - \boldsymbol{\Omega}\times\boldsymbol{c}\right)\mathrm{d}t + \mathrm{d}\boldsymbol{r}\cdot\boldsymbol{e}\right\}\cdot\frac{\partial \boldsymbol{a}}{\partial \boldsymbol{w}}.$$

Thus

$$\mathrm{d}_0\boldsymbol{A} = \mathrm{d}\boldsymbol{A} - \boldsymbol{\Omega}\times\boldsymbol{A}\,\mathrm{d}t = \left(\frac{\partial \boldsymbol{A}}{\partial t} - \boldsymbol{\Omega}\times\boldsymbol{A}\right)\mathrm{d}t + \mathrm{d}\boldsymbol{r}\cdot\frac{\partial \boldsymbol{A}}{\partial \boldsymbol{r}} + \mathrm{d}\boldsymbol{c}\cdot\frac{\partial \boldsymbol{A}}{\partial \boldsymbol{c}}$$

equals $\mathrm{d}\boldsymbol{a}$ provided that

$$\left.\begin{aligned} &\frac{\partial \boldsymbol{a}}{\partial \boldsymbol{w}} = \frac{\partial \boldsymbol{A}}{\partial \boldsymbol{c}}, \quad \frac{\partial \boldsymbol{a}}{\partial \boldsymbol{r}} = \frac{\partial \boldsymbol{A}}{\partial \boldsymbol{r}} - \boldsymbol{e}\cdot\frac{\partial \boldsymbol{A}}{\partial \boldsymbol{c}}, \\ &\frac{\partial \boldsymbol{a}}{\partial t} = \frac{\partial \boldsymbol{A}}{\partial t} - \boldsymbol{\Omega}\times\boldsymbol{A} - \left(\frac{\partial \boldsymbol{v}}{\partial t} - \boldsymbol{\Omega}\times\boldsymbol{c}\right)\cdot\frac{\partial \boldsymbol{A}}{\partial \boldsymbol{c}}. \end{aligned}\right\} \tag{5.3}$$

It follows that

$$\frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t} = \frac{\partial \boldsymbol{a}}{\partial t} + \boldsymbol{w}\cdot\frac{\partial \boldsymbol{a}}{\partial \boldsymbol{r}} + \boldsymbol{F}\cdot\frac{\partial \boldsymbol{a}}{\partial \boldsymbol{w}}$$

transforms into

$$\frac{\mathrm{d}_0 A}{\mathrm{d}t} = \mathrm{D}^* A + \boldsymbol{c}\cdot\boldsymbol{\nabla} A + (\boldsymbol{F} - \mathrm{D}\boldsymbol{v} - \boldsymbol{c}\cdot\boldsymbol{e}^{\mathrm{s}})\cdot\frac{\partial A}{\partial \boldsymbol{c}}, \tag{5.4}$$

since

$$\boldsymbol{c}\cdot\boldsymbol{e} - \boldsymbol{\Omega}\times\boldsymbol{c} = \boldsymbol{c}\cdot(\boldsymbol{e} + \boldsymbol{I}\times\boldsymbol{\Omega}) = \boldsymbol{c}\cdot(\boldsymbol{e} + \tfrac{1}{2}(\tilde{\boldsymbol{e}} - \boldsymbol{e})) = \boldsymbol{c}\cdot\boldsymbol{e}^{\mathrm{s}}$$

where $\boldsymbol{e}^{\mathrm{s}}$ is the symmetric part of $\boldsymbol{e}$, and D^*A denotes the frame-indifferent time derivative

$$\mathrm{D}^* A = \mathrm{D}A - \boldsymbol{\Omega}\times A, \quad \mathrm{D} \equiv \frac{\partial}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}. \tag{5.5}$$

The result contained in (5.4) is readily generalized to the following. Let Ψ represent a scalar Ψ_{s}, a vector $\boldsymbol{\Psi}_{\mathrm{v}}$ or a symmetric tensor $\boldsymbol{\Psi}_{\mathrm{t}}$, then its rate of change $\mathscr{D}^*\Psi$ in the O-frame equals its rate of change $\mathrm{d}\Psi/\mathrm{d}t$ in the L-frame, provided that

$$\mathscr{D}^*\Psi = \mathrm{D}^*\Psi + \boldsymbol{c}\cdot\boldsymbol{\nabla}\Psi + (\boldsymbol{F} - \mathrm{D}\boldsymbol{v} - \boldsymbol{c}\cdot\boldsymbol{e}^{\mathrm{s}})\cdot\frac{\partial \Psi}{\partial \boldsymbol{c}}, \tag{5.6a}$$

where

$$\mathrm{D}^*\Psi_{\mathrm{s}} = \mathrm{D}\Psi_{\mathrm{s}}, \quad \mathrm{D}^*\boldsymbol{\Psi}_{\mathrm{v}} = \mathrm{D}\boldsymbol{\Psi}_{\mathrm{v}} - \boldsymbol{\Omega}\times\boldsymbol{\Psi}_{\mathrm{v}}, \quad \mathrm{D}^*\boldsymbol{\Psi}_{\mathrm{t}} = \mathrm{D}\boldsymbol{\Psi}_{\mathrm{t}} - 2\boldsymbol{\Omega}\times\boldsymbol{\Psi}_{\mathrm{t}}. \tag{5.6b}$$

The operator $\mathscr{D}^*$ is a frame-indifferent time derivative – see the remark following (3.17).

We are now able to correct the theory of §§2 and 3 to a frame-indifferent form by the simple modification of replacing the frame-dependent operator $\mathscr{D}$ defined in (2.5) by $\mathscr{D}^*$. It will be sufficient to list a few of the important changes.

For (2.11) we now have

$$\mathscr{D}_0^* = \mathrm{D}_0^* + \boldsymbol{c}\cdot\boldsymbol{\nabla} + \left(\frac{\boldsymbol{\nabla}p}{\rho} - \boldsymbol{c}\cdot\boldsymbol{e}^{\mathrm{s}}\right)\cdot\frac{\partial}{\partial \boldsymbol{c}}, \quad \mathscr{D}_r^* = \mathscr{D}_r \quad (r \geqslant 1),$$

(2.14a) becomes

$$\mathscr{D}_0^*\,\boldsymbol{w} = \tfrac{1}{2}\boldsymbol{g} - \tfrac{1}{2}\boldsymbol{w}\boldsymbol{w}\cdot\boldsymbol{h} - \boldsymbol{w}\cdot\overset{0}{\boldsymbol{e}}, \quad \overset{0}{\boldsymbol{e}} = \boldsymbol{e}^{\mathrm{s}} - \tfrac{1}{3}\overset{\times}{e}\boldsymbol{I}.$$

The first-order theory in §3 is unchanged. For (3.6) and (3.10) we have

$$\phi_2 = -\tau\{\mathscr{D}_0^*\phi_1 + \phi_1\mathscr{D}_0^*\ln f_0 + \mathscr{D}_1\ln f_0\},$$

$$\mathscr{D}_0^*\boldsymbol{h} = \mathrm{D}_0^*\boldsymbol{h} + \boldsymbol{c}\cdot\boldsymbol{\nabla}\boldsymbol{h}, \quad \mathscr{D}_0^*\overset{0}{\boldsymbol{e}} = \mathrm{D}_0^*\overset{0}{\boldsymbol{e}} + \boldsymbol{c}\cdot\boldsymbol{\nabla}\overset{0}{\boldsymbol{e}}.$$

The effect on (3.11) is to replace $\mathrm{D}\boldsymbol{\nabla}T - \boldsymbol{e}\cdot\boldsymbol{\nabla}T$ by $\mathrm{D}^*\boldsymbol{\nabla}T - \overset{0}{\boldsymbol{e}}\cdot\boldsymbol{\nabla}T - \tfrac{1}{3}\overset{\times}{e}\boldsymbol{\nabla}T$, and $\mathrm{D}\overset{0}{\boldsymbol{e}} - 2\boldsymbol{e}\cdot\overset{0}{\boldsymbol{e}}$ by $\mathrm{D}^*\overset{0}{\boldsymbol{e}} - 2\overset{0}{\boldsymbol{e}}\cdot\overset{0}{\boldsymbol{e}} - \tfrac{2}{3}\overset{\times}{e}\overset{0}{\boldsymbol{e}}$, with the consequence that the Burnett equations (3.12) and (3.13) are corrected to

$$\boldsymbol{q}_2 = \frac{k}{m}p\tau_1^2\{\theta_1\boldsymbol{\nabla}\cdot\boldsymbol{v}\boldsymbol{\nabla}T + \theta_2(\mathrm{D}\boldsymbol{\nabla}T - \boldsymbol{\Omega}\times\boldsymbol{\nabla}T) + \theta_3\frac{T}{p}\boldsymbol{\nabla}p\cdot\overset{0}{\boldsymbol{e}} + 3\theta_5\boldsymbol{\nabla}T\cdot\overset{0}{\boldsymbol{e}}\}, \tag{5.7}$$

with

$$\theta_1 = \tfrac{15}{4}(3-s), \quad \theta_2 = \tfrac{45}{8}, \quad \theta_3 = -3, \quad \theta_4 = 3, \quad \theta_5 = \tfrac{55}{8} + s,$$

$$\boldsymbol{\pi}_2 = p\tau_1^2\left\{\varpi_1\boldsymbol{\nabla}\cdot\boldsymbol{v}\overset{0}{\boldsymbol{e}} + \varpi_2(\mathrm{D}\overset{0}{\boldsymbol{e}} - \overset{0}{\overline{2\boldsymbol{\Omega}\times\overset{0}{\boldsymbol{e}}}}) + \varpi_3\frac{k}{m}\overset{0}{\overline{\boldsymbol{\nabla}\boldsymbol{\nabla}T}} + \varpi_4\frac{1}{pT}\overset{0}{\overline{\boldsymbol{\nabla}p\boldsymbol{\nabla}T}}\right.$$
$$\left. + \varpi_5\frac{k}{mT}\overset{0}{\overline{\boldsymbol{\nabla}T\boldsymbol{\nabla}T}} + \varpi_6\overset{0}{\overline{\overset{0}{\boldsymbol{e}}\cdot\overset{0}{\boldsymbol{e}}}}\right\}, \tag{5.8}$$

with

$$\varpi_1 = \tfrac{4}{3}(3-s), \quad \varpi_2 = 2, \quad \varpi_3 = 3, \quad \varpi_4 = 0, \quad \varpi_5 = 3s, \quad \varpi_6 = 4.$$

6. Conclusions

Returning to the discussion in the last paragraph of §4, the author now believes Burnett's equations to be flawed, not because of an error intrinsic to the Chapman–Enskog theory, but simply because of an inadequate definition of what is meant by the 'peculiar velocity', an imprecision that also vitiates the physical arguments supporting Burnett. Of course one could take the view that it is merely a question of definition, e.g. that $\boldsymbol{p}$ and $\boldsymbol{p}'$ in (3.17),

$$\boldsymbol{p}' = \boldsymbol{p} + 4p\tau_1^2\varpi_2\,\omega \times \overset{0}{\boldsymbol{e}}, \tag{6.1}$$

specified in L and L' respectively, correctly represent different pressure tensors. In such a model, 'pressure tensor' does not refer to an objective property, *viz* one independent of the observer's motion, but, provided that its definition is kept in mind, no error need result. However, it is certainly preferable to describe reputed 'material' properties objectively if possible; further, as the dissipation rate D in a system is certainly objective once the observer's timescale has been decided, the constitutive relations must at least render D frame-indifferent.

Constitutive relations in which the vorticity $2\boldsymbol{\Omega}$ plays a leading role are not necessarily frame-dependent, even if not combined with a time derivative as in (5.6*b*). Rotating superfluids provide a well-known example (see e.g. Woods 1975), with the vorticity quantized into vortex lines, the strength of each line being frame-indifferent. Another example occurs in suspension mechanics (Ryskin 1980), where the rotational motion will change the structure of the microscale fluid motion around a particle, perhaps to the extent of generating a Taylor column. But as the microflow is objective, generating the same dissipation for all observers, it should be possible to specify the associated constitutive equation in a frame-indifferent form. In this example the vorticity $2\boldsymbol{\Omega}$ appearing in D is measured in the laboratory frame L (defined by the absence of rotational effects), and hence has a value independent of the observer's motion.

In an appendix to Ryskin's (1980) paper, Ryskin and Rallison make the point that observers may be unable to avoid giving a frame-dependent description of what they admit should be an objective property, e.g. they may not be able to find the inertial frame in which $\boldsymbol{\Omega}$ is to be measured. (As if with (6.1) the observer in L appreciates that $\boldsymbol{p}$ should be objective, but has no way of finding the correct frame in which to define $\boldsymbol{p}$.) However this may be, when frame-dependent constitutive equations are used to evaluate D, if this rate is not frame-indifferent, the model is wrong.

To sum up, in the author's view, while frame-indifference is a *fundamental* principle only for the dissipation rate, it is a convenient and desirable property for constitutive relations that can sometimes be achieved by taking care with definitions.

Professor A. E. Green and I recently debated several axioms of continuum mechanics (Woods 1981, 1982*b*; Green 1982), an important one being material frame-indifference. I am pleased to acknowledge that for this axiom his convictions were sound, whereas my confidence in the opposing evidence from kinetic theory etc. was misplaced.

REFERENCES

BURNETT, D. 1935 *Proc. Lond. Math. Soc.* **39**, 385–430; **40**, 382–435.

CHAPMAN, S. & COWLING, T. G. 1970 *The Mathematical Theory of Non-Uniform Gases.* Oxford University Press.

EDELEN, D. G. B. & MCLENNAN, J. A. 1973 *Intl J. Engng Sci.* **11**, 813.

ERINGEN, A. C. 1962 *Non-Linear Theory of Continuous Media.* McGraw-Hill.

FERTZIGER, J. H. & KAPER, H. G. 1972 *Mathematical Theory of Transport Processes in Gases.* North-Holland.

GREEN, A. E. 1982 *Bull. IMA* **18**, 7–9, 154.

IKENBERRY, E. & TRUESDELL, C. 1956 *J. Rat. Mech. Anal.* **5**, 1.

MÜLLER, I. 1972 *Arch. Rat. Mech. Anal.* **45**, 241–250.

RYSKIN, G. 1980 *J. Fluid Mech.* **99**, 513–529.

SPEZIALE, C. G. 1981 *Intl J. Engng Sci.* **19**, 63–73.

SÖDERHOLM, L. H. 1976 *Intl J. Engng. Sci.* **14**, 523.

TRUESDELL, C. & MUNCASTER, R. G. 1980 *Fundamentals of Maxwell's Kinetic Theory of a Simple Monatomic Gas.* Academic.

TRUESDELL, C. & NOLL, X. 1965 *The Non-Linear Field Theories of Mechanics. Handbuch der Physik* (ed. S. Flugge), vol. III/3. Springer.

WANG, C.-C. 1975 *Arch. Rat. Mech. Anal.* **58**, 381.

WANG, C.-C. 1976 *Q. Appl. Maths* **33**, 369.

WOODS, L. C. 1975 *The Thermodynamics of Fluid Systems.* Oxford University Press.

WOODS, L. C. 1979 *J. Fluid Mech.* **93**, 585–607.

WOODS, L. C. 1980 *J. Fluid Mech.* **101**, 225–241.

WOODS, L. C. 1981 *Bull. IMA* **17**, 98–102.

WOODS, L. C. 1982*a* *IMA J. Appl. Maths* **29**, 221–246.

WOODS, L. C. 1982*b* *Bull. IMA* **18**, 64–66.

WOODS, L. C. 1983 *Phys. Fluids* (submitted).